# Higher approximations in boundary-layer theory

## Part 1. General analysis

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(Received 5 April 1962)

Prandtl's boundary-layer theory is embedded as the first step in a systematic scheme of successive approximations for finding an asymptotic solution for viscous flow at large Reynolds number. The technique of inner and outer expansions is used to treat this singular-perturbation problem. Only analytic semi-infinite bodies free of separation are considered. The second approximation is analysed in detail for steady laminar flow past plane or axisymmetric solid bodies. Attention is restricted to low speeds and small temperature changes, so that the velocity field is that for an incompressible fluid, the temperature field being calculated subsequently. The additive effects are distinguished of longitudinal curvature, transverse curvature, external vorticity, external stagnation enthalpy gradient, and displacement speed. The effect of changing co-ordinates is examined, and the behaviour of the boundary-layer solution far downstream discussed. Application to specific problems will be made in subsequent papers.

## 1. Introduction

It has been recognized for some time that Prandtl's boundary-layer theory yields only the first term in an asymptotic solution of the Navier–Stokes equations for large Reynolds number. However, attempts to calculate further terms in the asymptotic expansion have been fragmentary. Prandtl himself (1935) suggested how the boundary layer on a flat plate might be corrected for the effect of displacement thickness. The effect of longitudinal curvature of the surface was analysed in a special case by Murphy (1953), and that of transverse curvature on a body of revolution by Seban & Bond (1951). The effect of external vorticity was pointed out by Ferri & Libby (1954). Ovchinnikov (1960) has considered the effect of an external gradient of total enthalpy.

Each of these is a secondary effect, of relative order $R^{-\frac{1}{2}}$, where $R$ is an appropriate Reynolds number. Although each has received further attention, the results remain isolated, unco-ordinated, sometimes erroneous, and in two cases the subject of prolonged controversies. The aim of the present work is to provide a unified theory of higher-order effects, with emphasis on the second approximation.

In a singular-perturbation problem of this sort, higher approximations are most efficiently found by the so-called method of inner and outer expansions. This technique has been extensively developed by Kaplun (1954), Lagerstrom & Cole (1955) and their colleagues at Caltech, and others. (See Lagerstrom 1957;

Erdelyi 1961.) Two complementary asymptotic expansions are constructed simultaneously, and matched in their overlap region of common validity. For viscous flows at low Reynolds number, this procedure has been discussed by Kaplun & Lagerstrom (1957), and applied to the circular cylinder and sphere by Kaplun (1957) and Proudman & Pearson (1957). The corresponding analysis for high Reynolds number is given here. The general theory is developed in the present Part 1, application being made to leading edges in Part 2, and to the parabolic cylinder in a uniform stream in Part 3.

## 2. Formulation of problem

### 2.1. *The fluid and body*

Consider steady laminar flow of a viscous fluid. For simplicity let the flow be incompressible. (The extension to compressible flow is outlined in Van Dyke 1962.) We adopt the broadest possible interpretation of this term, so that low-speed flow of a gas as well as of a liquid is admitted. We require only that the momentum and continuity equations be uncoupled from the energy equation. That is the case if the Mach number is low and the relative variation in temperature small, so that the heat added by dissipation and by conduction is slight. Then the density is sensibly constant, as are the transport properties. The velocity and pressure field is governed by the Navier–Stokes equations for an incompressible fluid. The temperature field can subsequently be calculated from the energy equation, which is linear with non-constant coefficients depending upon the velocity field.

For simplicity let the motion be either plane (but not necessarily symmetric) or axisymmetric, and the body solid. The appropriate solution of the inviscid (Euler) equations to describe the limiting flow at infinite Reynolds number is not known for separated flows, and has a complicated mathematical description for unseparated flow past finite bodies. Also, serious difficulties arise for non-analytic shapes (see Goldstein 1960, ch. 8). Therefore we consider only unseparated flow past a semi-infinite body described by an analytic curve. The surface temperature condition is also assumed analytic.

### 2.2. *Upstream conditions*

Suppose that a well-posed problem has been set for the Navier–Stokes equations, for which we seek the asymptotic form of the solution for large Reynolds number. Then the flow upstream is prescribed, and in general satisfies the viscous flow equations. This requirement is satisfied trivially in the classical case of a uniform parallel stream or other irrotational motion, because any potential flow satisfies the Navier–Stokes equations. However, if the oncoming stream contains vorticity, a solution of the Euler equations does not in general satisfy the Navier–Stokes equations. It does so, however, in special cases, including the useful ones of plane flow with constant vorticity and axisymmetric flow with vorticity proportional to the radius.

This general situation permits the oncoming stream (even if it is irrotational) to depend upon the Reynolds number. This would occur, for example, for a slender body in a wind tunnel whose free stream would vary slightly with the thickness of the wall boundary layers. However, for simplicity we assume that conditions far upstream are independent of Reynolds number. The oncoming stream is then represented by a solution of the inviscid equations.

### 2.3. *Dimensionless equations and boundary conditions*

It is convenient to refer all lengths to a typical body dimension $L$ (e.g. its nose radius), speeds and temperatures to characteristic reference values $U_r$ and $T_r$, and pressures to $\rho U_r^2$. Then under our assumption of constant fluid properties the Navier–Stokes equations, which govern the velocity and pressure field, become

$$\operatorname{div}\mathbf{q} = 0, \tag{2.1}$$

$$(\operatorname{grad}\mathbf{q}).\mathbf{q}+\operatorname{grad}p = -R^{-1}\operatorname{curl}(\operatorname{curl}\mathbf{q}). \tag{2.2}$$

The notation is standard, $R = U_r L\rho/\mu$ being the characteristic Reynolds number.

Variations in temperature can then be calculated from the energy equation with the fluid properties taken constant so that it is linear:

$$\mathbf{q}.\operatorname{grad}T-\frac{1}{\sigma R}\nabla^2 T = m^2\left[\frac{1}{R}\operatorname{grad}\mathbf{q}.\operatorname{def}\mathbf{q}-\left(\frac{\partial\log\rho}{\partial\log T}\right)_p\mathbf{q}.\operatorname{grad}p\right]. \tag{2.3}$$

Here $\sigma = \mu c_p/k$ is the Prandtl number, and $\operatorname{def}\mathbf{q}$ the deformation tensor, $\operatorname{grad}\mathbf{q}$ plus its transpose (Lagerstrom 1962). The factor $m^2$ is given by

$$m^2 = U_r^2/c_p T_r, \tag{2.4}$$

which is $(\gamma-1)M_r^2$ for a perfect gas, $M_r$ being the characteristic Mach number based upon $U_r$ and $T_r$. For a liquid the thermal expansion parameter

$$(\partial\log\rho/\partial\log T)_p$$

is normally small; in any case $m^2$ is so small that the right-hand side of (2.3) is ordinarily neglected. It will therefore be retained only for a perfect gas, for which we can henceforth set $(\partial\log\rho/\partial\log T)_p = -1$.

These equations are accurate if both $m^2$ and the imposed relative temperature difference $\Delta T$ are small. More precisely, a perturbation analysis (following Lagerstrom 1962) shows that they give the temperature correctly including terms of order $m^2$ and $\Delta T$, although the velocity and pressure are correct only to order unity.

An alternative form of the energy equation is preferable, which involves stagnation enthalpy in so far as possible. First forming the inner product of $\mathbf{q}$ and (2.2) gives the kinematic energy equation

$$\mathbf{q}.\operatorname{grad}(p+\tfrac{1}{2}q^2) = R^{-1}\mathbf{q}.\nabla^2\mathbf{q}. \tag{2.5}$$

Then adding $m^2$ times this to (2.3) yields

$$\mathbf{q}.\operatorname{grad}(T+\tfrac{1}{2}m^2q^2)-R^{-1}\nabla^2(\sigma^{-1}T+\tfrac{1}{2}m^2q^2) = m^2R^{-1}(\operatorname{grad}\mathbf{q}).(\operatorname{grad}\mathbf{q})^*, \tag{2.6}$$

where the asterisk denotes the transpose, with $m^2 = 0$ for a liquid.

The pressure can be eliminated from the momentum equation by taking its curl, which leaves an equation for the vorticity vector $\operatorname{curl}\mathbf{q}$. For plane or axisymmetric flow the vorticity has only a component of magnitude $\omega$ normal to the plane of flow. Then the vorticity equation reduces to

$$r^j\mathbf{q}.\mathrm{grad}\left(\frac{\omega}{r^j}\right) = \frac{1}{R}\mathrm{div}\left[\frac{1}{r^j}\mathrm{grad}\,(r^j\omega)\right], \tag{2.7}$$

where $r$ is the distance from the axis in axisymmetric flow, and $j = 0$ for plane flow and $j = 1$ for axisymmetric flow. Henceforth all vector operations are restricted to the plane of flow, and in particular div must here be the two-dimensional divergence operator. The two scalar equations (2.5) and (2.7) are equivalent to (2.2).

The boundary condition at a solid body is $\mathbf{q} = 0$ if slip is neglected. The condition on temperature may prescribe its surface value, require that its normal gradient vanish (for an insulated body), or the like. Far upstream the flow is to approach prescribed (possibly non-uniform) velocity and temperature fields $Q_\infty$ and $T_\infty$.

### 2.4. *Orthogonal co-ordinates and stream function*

In specific problems one usually introduces orthogonal curvilinear co-ordinates $(\xi, \eta)$ in the plane of flow, with corresponding velocity components $(u, v)$. The length element $dl$ in space is then given by

$$dl^2 = e_1^2(\xi,\eta)\,d\xi^2 + e_2^2(\xi,\eta)\,d\eta^2 + r^{2j}\,d\phi^2, \tag{2.8}$$

where $\phi$ is the third Cartesian co-ordinate for plane flow and the azimuthal angle for axisymmetric flow.

The preceding equations are readily expressed in these co-ordinates using standard vector relations. In particular, the continuity equation (2.1) becomes

$$\frac{\partial}{\partial\xi}(r^je_2u) + \frac{\partial}{\partial\eta}(r^je_1v) = 0. \tag{2.9}$$

This is satisfied by introducing the stream function $\psi$ in the usual way:

$$\partial\psi/\partial\eta = r^je_2u, \quad \partial\psi/\partial\xi = -r^je_1v. \tag{2.10}$$

Then the scalar vorticity is given by

$$\begin{aligned}\omega &= \frac{1}{e_1e_2}\left[\frac{\partial}{\partial\xi}(e_2v) - \frac{\partial}{\partial\eta}(e_1u)\right] \\ &= -\frac{1}{e_1e_2}\left[\frac{\partial}{\partial\xi}\left(\frac{e_2}{r^je_1}\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{e_1}{r^je_2}\frac{\partial\psi}{\partial\eta}\right)\right].\end{aligned} \tag{2.11}$$

We make particular use of the orthogonal co-ordinates $(s, n)$ indicated in figure 1. Here $n$ is the distance normal to the surface of the body, and $s$ the distance to the foot of the normal measured along the surface from, say, the stagnation point. (These co-ordinates are ambiguous away from the surface of a concave body, but that is of no concern because they will be used only arbitrarily near the surface.) Let $\kappa(s)$ be the curvature of the body in the plane of flow (positive for a convex shape) and, for use in axisymmetric flow, $\theta(s)$ the angle between the axis and the tangent to the meridian curve at any point and $r_0(s)$ the distance of the point from the axis. Any one of these functions determines the other two according to

$$\sin\theta = r_0', \quad \cos\theta = -\kappa r_0''. \tag{2.12}$$

Then (2.8) specializes to

$$dl^2 = (1+\kappa n)^2 ds^2 + dn^2 + (r_0 + n\cos\theta)^{2j} d\phi^2. \tag{2.13}$$

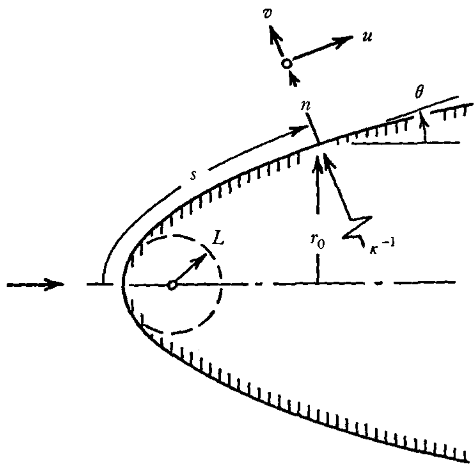


FIGURE 1. Co-ordinates for boundary layer.

## 3. Outer and inner expansions

### 3.1. *Outer expansion*

As the Reynolds number $R$ increases, the motion at any fixed point away from the surface approaches an inviscid flow. Kaplun (1954) and Lagerstrom & Cole (1955) call this the *Euler* or *outer limit*. Repeated application of the outer limit process, in conjunction with an appropriate sequence of functions of $R$, produces an asymptotic expansion for large $R$, the *outer expansion*. Under our restrictions (to an analytic semi-infinite body free of separation) the asymptotic sequence is believed to consist of negative half-powers of $R$. The outer expansion therefore has the form

$$\left.\begin{aligned} \mathbf{q}(\mathbf{x}; R) &\sim \mathbf{Q}_1(\mathbf{x}) + R^{-\frac{1}{2}}\mathbf{Q}_2(\mathbf{x}) + \ldots, \\ p(\mathbf{x}; R) &\sim P_1(\mathbf{x}) + R^{-\frac{1}{2}}P_2(\mathbf{x}) + \ldots, \\ T(\mathbf{x}; R) &\sim T_1(\mathbf{x}) + R^{-\frac{1}{2}}T_2(\mathbf{x}) + \ldots. \end{aligned}\right\} \tag{3.1}$$

Likewise, the components of $\mathbf{q}$ in any co-ordinate system, the vorticity, and the stream function have the outer expansions

$$\left.\begin{aligned} u(\mathbf{x}; R) &\sim U_1(\mathbf{x}) + R^{-\frac{1}{2}}U_2(\mathbf{x}) + \ldots, \\ v(\mathbf{x}; R) &\sim V_1(\mathbf{x}) + R^{-\frac{1}{2}}V_2(\mathbf{x}) + \ldots, \\ \omega(\mathbf{x}; R) &\sim \Omega_1(\mathbf{x}) + R^{-\frac{1}{2}}\Omega_2(\mathbf{x}) + \ldots, \\ \psi(\mathbf{x}; R) &\sim \Psi_1(\mathbf{x}) + R^{-\frac{1}{2}}\Psi_2(\mathbf{x}) + \ldots. \end{aligned}\right\} \tag{3.2}$$

It is implied that the functions $\mathbf{Q}_1$, $\mathbf{Q}_2$, etc. and their derivatives are all of order unity.

Substituting these expansions into the equations of motion and equating like powers of $R$ yields equations for the successive outer approximations. Thus (2.1), (2.2) and (2.6) give for the first approximation

$$\operatorname{div} \mathbf{Q}_1 = 0, \tag{3.3a}$$

$$\mathbf{Q}_1 . \operatorname{grad} \mathbf{Q}_1 + \operatorname{grad} P_1 = 0, \tag{3.3b}$$

$$\mathbf{Q}_1 . \operatorname{grad} (T_1 + \tfrac{1}{2}m^2 Q_1^2) = 0, \tag{3.3c}$$

which are the Euler equations for the basic inviscid flow, and for the second approximation

$$\operatorname{div} \mathbf{Q}_2 = 0, \tag{3.4a}$$

$$\mathbf{Q}_1 . \operatorname{grad} \mathbf{Q}_2 + \mathbf{Q}_2 . \operatorname{grad} \mathbf{Q}_1 + \operatorname{grad} P_2 = 0, \tag{3.4b}$$

$$\mathbf{Q}_1 . \operatorname{grad} (T_2 + m^2 \mathbf{Q}_1 . \mathbf{Q}_2) + \mathbf{Q}_2 . \operatorname{grad} (T_1 + \tfrac{1}{2}m^2 Q_1^2) = 0, \tag{3.4c}$$

which are the small-perturbation form of the Euler equations. Viscous terms appear in the outer equations beginning only with the third approximation.

The outer expansion is invalid at the body surface, where the no-slip condition must be given up. In fact, it appears that all boundary conditions at the body must be dropped except that on the normal component of velocity in the first approximation:

$$V_1 = 0 \quad \text{at} \quad n = 0. \tag{3.5}$$

### 3.2. *Integrals of outer equations*

Substituting the outer expansions into the kinematic energy equation (2.5) and vorticity equation (2.7) gives as an alternative to the first-order momentum equation (3.3*b*)

$$\mathbf{Q}_1 . \operatorname{grad} (P_1 + \tfrac{1}{2}Q_1^2) = 0, \tag{3.6}$$

$$\mathbf{Q}_1 . \operatorname{grad} (\Omega_1 / r^j) = 0. \tag{3.7}$$

Each of (3.3*c*), (3.6) and (3.7) expresses conservation along streamlines of a flow quantity, which is therefore a function only of $\Psi_1$. The three functions are not independent, however, being connected by the Bernoulli equation (which is a consequence of the Euler equations (3.3)):

$$\Omega_1 / r^j = -d(P_1 + \tfrac{1}{2}Q_1^2)/d\Psi_1. \tag{3.8}$$

Hence the first-order outer problem has the three first integrals

$$P_1 + \tfrac{1}{2}Q_1^2 = B_1(\Psi_1), \tag{3.9a}$$

$$\Omega_1 / r^j = -B_1'(\Psi_1), \tag{3.9b}$$

$$T_1 + \tfrac{1}{2}m^2 Q_1^2 = H_1(\Psi_1). \tag{3.9c}$$

The Bernoulli function $B_1$ and stagnation enthalpy function $H_1$ are to be evaluated from the upstream conditions.

Similarly, for the second approximation one obtains the perturbation forms of (3.6) and (3.7), and the first integrals

$$P_2 + \mathbf{Q}_1 . \mathbf{Q}_2 = \Psi_2 B_1'(\Psi_1) + B_2(\Psi_1), \tag{3.10a}$$

$$\Omega_2 / r^j = -\Psi_2 B_1''(\Psi_1) - B_2'(\Psi_1), \tag{3.10b}$$

$$T_2 + m^2 \mathbf{Q}_1 . \mathbf{Q}_2 = \Psi_2 H_1'(\Psi_1) + H_2(\Psi_1). \tag{3.10c}$$

The first terms on the right express (through Taylor series expansion about the basic inviscid flow) the facts that stagnation pressure and enthalpy are actually conserved along perturbed rather than basic streamlines, and the second terms allow for small variations of those quantities with Reynolds number. However, we have assumed for simplicity that the oncoming stream is independent of Reynolds number, which means that we take $B_2 = H_2 = 0$.

### 3.3 *First and second outer problems*

One usually solves the outer problem using the stream function in orthogonal co-ordinates. An equation for $\Psi_1$ alone is obtained from (3.9*b*) with the aid of (2.11). Thus the problem for the basic inviscid flow becomes

$$\left[\frac{\partial}{\partial\xi}\left(\frac{e_2}{r^j e_1}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{e_1}{r^j e_2}\frac{\partial}{\partial\eta}\right)\right]\Psi_1 = r^j e_1 e_2 B_1'(\Psi_1), \tag{3.11a}$$

$$\Psi_1 = 0 \quad \text{on the body}, \tag{3.11b}$$

$$\Psi_1 \sim \Psi_{1\infty}(\xi,\eta) \quad \text{upstream}, \tag{3.11c}$$

where $B_1$ and $\Psi_{1\infty}$ are known functions. This is a properly set problem for an elliptic differential equation. The equation is linear if $B_1$ is a linear function of its argument. The solution for $\Psi_1$ provides the velocity field, and the pressure and temperature are then given by the integrals (3.9*a*) and (3.9*c*).

For the second approximation (3.10*b*) gives, with $B_2 = 0$, the differential equation

$$\left[\frac{\partial}{\partial\xi}\left(\frac{e_2}{r^j e_1}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{e_1}{r^j e_2}\frac{\partial}{\partial\eta}\right)\right]\Psi_2 = r^j e_1 e_2 B_1''(\Psi_1)\,\Psi_2. \tag{3.12}$$

This is also elliptic but, as usual in perturbation schemes, linear with coefficients depending on the basic flow. The only applicable boundary condition is that the velocity due to $\Psi_2$ must vanish upstream. The necessary additional boundary condition (3.27*b*) will be found by matching with the inner solution. The pressure and temperature perturbations are found from (3.10*a*) and (3.10*c*).

### 3.4. *Outer flow near surface*

In classical boundary-layer theory the outer flow enters only through the values of velocity (or pressure) and temperature that it predicts at the surface. In higher approximations, normal derivatives of the outer solution at the surface are also required. We consider the quantities needed for the second approximation.

We use henceforth the co-ordinates $(s, n)$ of figure 1. The basic inviscid flow provides the surface speed $U_1(s,0)$, and then (3.9*a*) and (3.9*c*) give

$$P_1(s,0) = B_1(0) - \tfrac{1}{2}U_1^2(s,0) \tag{3.13a}$$

$$T_1(s,0) = H_1(0) - \tfrac{1}{2}m^2 U_1^2(s,0). \tag{3.13b}$$

Matching will be used later to determine $V_2$ and $\Psi_2$ at the surface in terms of the first-order boundary-layer solution. Then the second-order outer solution provides $U_2(s,0)$, and (3.10*a*) and (3.10*c*) give

$$P_2(s,0) = B_1'(0)\,\Psi_2(s,0) - U_1(s,0)\,U_2(s,0), \tag{3.14a}$$

$$T_2(s,0) = H_1'(0)\,\Psi_2(s,0) - m^2 U_1(s,0)\,U_2(s,0). \tag{3.14b}$$

Writing (3.3$a$) and (3.3$b$) in the $(s, n)$ co-ordinates and evaluating them at the surface gives two of the required normal derivatives:

$$V_{1n}(s,0) = -r_0^{-j} d[r_0^j U_1(s,0)]/ds, \tag{3.15a}$$

$$P_{1n}(s,0) = \kappa U_1^2(s,0). \tag{3.15b}$$

The other two are found by differentiating (3.9$a$) and (3.9$c$) and using (3.15$b$) and the fact that $\Psi_{1n}(s,0) = r_0^j U_1(s,0)$:

$$U_{1n}(s,0) = B_1'(0)\, r_0^j - \kappa U_1(s,0), \tag{3.16a}$$

$$T_{1n}(s,0) = U_1(s,0)\,[H_1'(0)\, r_0^j - B_1'(0)\, m^2 r_0^j + m^2 \kappa U_1(s,0)]. \tag{3.16b}$$

### 3.5. *Inner expansion*

The outer expansion violates the conditions on velocity and temperature at the wall. It is therefore invalid within the boundary layer, whose dimensionless thickness is (for finite $\sigma$ and $s$) of order $R^{-\frac{1}{2}}$. Following Prandtl, we magnify the normal co-ordinate accordingly by introducing the boundary-layer variable

$$N = R^{\frac{1}{2}} n. \tag{3.17}$$

The normal velocity and stream function are likewise small in the boundary layer, and must be magnified similarly. Then Prandtl's boundary-layer approximation is obtained by letting $R$ become infinite with the *inner variables* $s$ and $N$ fixed. Repeated application of this *inner* or *Prandtl limit*, in conjunction with an appropriate sequence of functions of $R$, produces the *inner expansion*. For an analytic semi-infinite body free of separation this sequence is believed to consist again of negative half powers. Thus the inner expansion is

$$\left.\begin{aligned} u(s,n;R) &\sim u_1(s,N) + R^{-\frac{1}{2}} u_2(s,N) + \ldots, \\ v(s,n;R) &\sim R^{-\frac{1}{2}} v_1(s,N) + R^{-1} v_2(s,N) + \ldots, \\ p(s,n;R) &\sim p_1(s,N) + R^{-\frac{1}{2}} p_2(s,N) + \ldots, \\ T(s,n;R) &\sim t_1(s,N) + R^{-\frac{1}{2}} t_2(s,N) + \ldots. \end{aligned}\right\} \tag{3.18}$$

Later we shall use also the inner expansion for the stream function

$$\psi(s,n;R) \sim R^{-\frac{1}{2}} \psi_1(s,N) + R^{-1} \psi_2(s,N) + \ldots. \tag{3.19}$$

Substituting into (2.1), (2.2) and (2.6) gives for the first approximation the equations

$$(r_0^j u_1)_s + (r_0^j v_1)_N = 0, \tag{3.20a}$$

$$u_1 u_{1s} + v_1 u_{1N} + p_{1s} - u_{1NN} = 0, \tag{3.20b}$$

$$p_{1N} = 0, \tag{3.20c}$$

$$\left(u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N}\right)(t_1 + \tfrac{1}{2} m^2 u_1^2) - \frac{\partial^2}{\partial N^2}(\sigma^{-1} t_1 + \tfrac{1}{2} m^2 u_1^2) = 0, \tag{3.20d}$$

where letter subscripts indicate differentiation, and for the second approximation

$$(r_0^j u_2)_s + (r_0^j v_2)_N = -[r_0^j(j\cos\theta/r_0)\,Nu_1]_s - [r_0^j(\kappa + j\cos\theta/r_0)\,Nv_1]_N, \tag{3.21a}$$

$$u_1 u_{2s} + u_2 u_{1s} + v_1 u_{2N} + v_2 u_{1N} + p_{2s} - u_{2NN}$$
$$= \kappa(Nu_{1NN} + u_{1N} - Nv_1 u_{1N} - u_1 v_1) + (j\cos\theta/r_0)\,u_{1N}, \tag{3.21b}$$

$$p_{2N} = \kappa u_1^2, \tag{3.21c}$$

$$\left(u_1\frac{\partial}{\partial s} + v_1\frac{\partial}{\partial_N}\right)(t_2 + m^2 u_1 u_2) + \left(u_2\frac{\partial}{\partial s} + v_2\frac{\partial}{\partial N}\right)(t_1 + \tfrac{1}{2}m^2u_1^2) - \frac{\partial^2}{\partial N^2}(\sigma^{-1}t_2 + m^2 u_1 u_2)$$
$$= \kappa N u_1\frac{\partial}{\partial s}(t_1 + \tfrac{1}{2}m^2u_1^2) + \kappa\frac{\partial}{\partial N}(\sigma^{-1}t_1 - \tfrac{1}{2}m^2u_1^2) + (j\cos\theta/r_0)\frac{\partial}{\partial N}(\sigma^{-1}t_1 + \tfrac{1}{2}m^2u_1^2). \tag{3.21d}$$

(Alternative forms of the non-homogeneous right-hand sides are related through the first-order equations.)

The requirement of zero velocity at the surface gives the boundary conditions

$$u_1(s,0) = v_1(s,0) = u_2(s,0) = v_2(s,0) = 0. \tag{3.22}$$

One condition is to be imposed on the temperature at the body; for example, prescribing its (dimensionless) value as $T_w(s)$ gives

$$t_1(s,0) = T_w(s), \quad t_2(s,0) = 0, \tag{3.23a}$$

whereas requiring an insulated surface gives

$$t_{1N}(s,0) = t_{2N}(s,0) = 0, \tag{3.23b}$$

etc. The upstream conditions will not in general be satisfied by the inner expansion.

### 3.6. *Matching conditions*

Insufficient boundary conditions are available except for the first outer problem. The missing conditions are supplied by matching the inner and outer expansions. Whereas at low Reynolds numbers a generalized matching principle is required, here it suffices to apply the restricted matching principle (Lagerstrom 1957)

$m$-term inner expansion of ($p$-term outer expansion)

= $p$-term outer expansion of ($m$-term inner expansion). (3.24)

Matching conditions for the classical boundary layer are found by taking $m = p = 1$. The one-term outer expansion of $u$ is $U_1(s,n)$; rewriting it in inner variables gives $U_1(s, R^{-\frac{1}{2}}N)$; and expanding for large $R$ (assuming analyticity) gives $U_1(s,0)$ as its one-term inner expansion. Conversely, one term of the inner expansion is $u_1(s,N)$; rewriting in outer variables gives $u_1(s, R^{\frac{1}{2}}n)$; and expanding for large $R$ gives $u_1(s,\infty)$ as its one-term outer expansion. Equating these results, and proceeding similarly with $p$ and $T$ gives the matching conditions

$$u_1(s,\infty) = U_1(s,0), \tag{3.25a}$$

$$p_1(s,\infty) = P_1(s,0), \tag{3.25b}$$

$$t_1(s,\infty) = T_1(s,0). \tag{3.25c}$$

These are recognized as the classical conditions that at the outer edge of the boundary layer the tangential velocity, pressure, and temperature approach the inviscid surface values.

Matching $v$ in the same way gives

$$v_{1N}(s,\infty) = -r_0^{-j}d[r_0^j U_1(s,0]/ds, \tag{3.26}$$

which is superfluous because it follows from (3.25$a$) by continuity. However, a useful result is obtained by taking instead $m=2$, $p=1$, which gives

$$V_2(s,0) = \lim_{N\to\infty}(v_1 - Nv_{1N}). \tag{3.27a}$$

The quantity on the right can be evaluated from the classical boundary-layer solution, being related to its displacement thickness. This is a matching condition for the second-order outer flow, which is often called the flow due to displacement thickness. An equivalent condition on the stream function is obtained using $r_0^j V_2(s,0) = -d\Psi_2(s,0)/ds$, or directly by matching $\psi$ in the same way:

$$\Psi_2(s,0) = -\lim_{N\to\infty}(N\psi_{1N} - \psi_1). \tag{3.27b}$$

These both have the physical interpretation that the displacement effect of the boundary layer upon the outer flow is that of a surface distribution of sources; this appears to be more fundamental than the concept of a displacement thickness.

Finally, matching conditions for the second-order boundary-layer problem are found by taking $m=p=2$ in (3.24). Using also (3.15) and (3.16) leads to

$$u_2(s,N) \sim N[B_1'(0)r_0^j - \kappa U_1(s,0)] + U_2(s,0), \tag{3.28a}$$

$$p_2(s,N) \sim N\kappa U_1^2(s,0) + P_2(s,0), \tag{3.28b}$$

$$t_2(s,N) \sim NU_1(s,0)[H_1'(0)r_0^j - m^2B_1'(0)r_0^j + m^2\kappa U_1(s,0)] + T_2(s,0), \tag{3.28c}$$

as $N\to\infty$, where $P_2(s,0)$ and $T_2(s,0)$ are given by (3.14). The remainders are known to be exponentially small, so that the asymptotic forms (3.28), like their first-order counterparts (3.15), are in fact approached more rapidly than any power of $N^{-1}$ (though this is not essential to the matching).

## 4. First- and second-order boundary-layer problems

### 4.1. *First-order boundary layer*

Equation (3.20$c$) shows that $p_1$ is constant across the boundary layer, and (3.25$b$) gives its value as $P_1(s,0)$, which is known from (3.13$a$). Thus (3.20) become the classical boundary-layer equations:

$$(r_0^j u_1)_s + (r_0^j v_1)_N = 0, \tag{4.1a}$$

$$u_1 u_{1s} + v_1 u_{1N} - u_{1NN} = U_1(s,0)\,U_{1s}(s,0), \tag{4.1b}$$

$$\left(u_1\frac{\partial}{\partial s} + v_1\frac{\partial}{\partial N}\right)(t_1 + \tfrac{1}{2}m^2u_1^2) - \frac{\partial^2}{\partial N^2}(\sigma^{-1}t_1 + \tfrac{1}{2}m^2u_1^2) = 0. \tag{4.1c}$$

The corresponding boundary conditions are given by (3.22), (3.23), (3.25) and (3.13*b*) as

$$u_1(s,0) = v_1(s,0) = 0, \tag{4.2a}$$

$$t_1(s,0) = T_w(s), \quad t_{1N}(s,0) = 0, \quad \text{or the like,} \tag{4.2b}$$

$$u_1(s,\infty) = U_1(s,0), \quad t_1(s,\infty) = H_1(0) - \tfrac{1}{2}m^2U_1^2(s,0). \tag{4.2c}$$

For an insulated wall and Prandtl number unity the energy equation (4.1*c*) has the Busemann–Crocco integral

$$t_1 + \tfrac{1}{2}m^2u_1^2 = H_1(0). \tag{4.3}$$

### 4.2. *Second-order boundary layer*

Integrating the normal-momentum equation (3.21*c*) with respect to $N$ and using the matching condition (3.28*b*) to evaluate the function of integration gives the increment in pressure within the boundary layer as

$$p_2(s,N) = N\kappa U_1^2(s,0) + \kappa\int_N^\infty \{U_1^2(s,0) - u_1^2(s,N)\}\,dN + P_2(s,0), \tag{4.4}$$

where $P_2(s,0)$ is given by (3.14*a*). Hence the second-order pressure gradient that appears in (3.21*b*) is

$$p_{2s}(s,N) = \frac{\partial}{\partial s}\kappa\left[NU_1^2(s,0) + \int_N^\infty \{U_1^2(s,0) - u_1^2(s,N)\}\,dN\right]$$
$$- B_1'(0)\,r_0^j V_2(s,0) - \frac{d}{ds}U_1(s,0)\,U_2(s,0). \tag{4.5}$$

The three terms on the right are the result of longitudinal curvature, interaction of displacement and external vorticity, and change in inviscid surface speed due to displacement.

Substituting (4.5) into (3.21) gives the second-order boundary-layer equations:

$$(r_0^j u_2)_s + (r_0^j v_2)_N = -[r_0^j(j\cos\theta/r_0)\,Nu_1]_s - [r_0^j(\kappa + j\cos\theta/r_0)\,Nv_1]_N, \tag{4.6a}$$

$$u_1u_{2s} + u_2u_{1s} + v_1u_{2N} + v_2u_{1N} - u_{2NN} = \kappa(Nu_{1NN} + u_{1N} - Nv_1u_{1N} - u_1v_1)$$
$$-\frac{\partial}{\partial s}\kappa\left[NU_1^2(s,0) + \int_N^\infty \{U_1^2(s,0) - u_1^2(s,N)\}\,dN\right] + (j\cos\theta/r_0)\,u_{1N}$$
$$+ B_1'(0)\,r_0^j V_2(s,0) + \frac{d}{ds}U_1(s,0)\,U_2(s,0), \tag{4.6b}$$

$$u_1t_{2s} + v_1t_{2N} - \sigma^{-1}t_{2NN} = \kappa Nu_1\frac{\partial}{\partial s}(t_1 + \tfrac{1}{2}m^2u_1^2) + \kappa\frac{\partial}{\partial N}(\sigma^{-1}t_1 - \tfrac{1}{2}m^2u_1^2)$$
$$-\left(u_2\frac{\partial}{\partial s} + v_2\frac{\partial}{\partial N}\right)(t_1 + \tfrac{1}{2}m^2u_1^2) - m^2\left(u_1\frac{\partial}{\partial s} + v_1\frac{\partial}{\partial N} - \frac{\partial^2}{\partial N^2}\right)u_1u_2$$
$$+ (j\cos\theta/r_0)\frac{\partial}{\partial N}(\sigma^{-1}t_1 + \tfrac{1}{2}m^2u_1^2). \tag{4.6c}$$

The boundary conditions are given by (3.22), (3.23) and (3.28) as

$$u_2(s,0) = v_2(s,0) = 0, \tag{4.7a}$$

$$t_2(s,0) = 0, \quad t_{2N}(s,0) = 0, \quad \text{or the like}, \tag{4.7b}$$

$$u_2(s,N) \sim N[B_1'(0)\, r_0^j - \kappa U_1(s,0)] + U_2(s,0) \quad \text{as} \quad N \to \infty, \tag{4.7c}$$

$$t_2(s,N) \sim NU_1(s,0)\,[H_1'(0)\, r_0^j - m^2 B_1'(0)\, r_0^j + m^2 \kappa U_1(s,0)] + H_1'(0)\,\Psi_2(s,0) - m^2 U_1(s,0)\, U_2(s,0). \tag{4.7d}$$

### 4.3. *Use of stream function*

In specific applications it is convenient to work with the stream function. The continuity equations (4.1*a*), (4.6*a*) are satisfied by introducing first- and second-order stream functions $\psi_1$, $\psi_2$ according to

$$r_0^j u_1 = \psi_{1N}, \quad r_0^j v_1 = -\psi_{1s}, \tag{4.8a}$$

$$r_0^j[u_2 + (j\cos\theta/r_0)\, Nu_1] = \psi_{2N}, \quad r_0^j[v_2 + (\kappa + j\cos\theta/r_0)\, Nv_1] = -\psi_{2s}, \tag{4.8b}$$

these being just the components of the inner expansion (3.19) for $\psi$. The first-order boundary-layer problem then becomes for the velocity field

$$\psi_{1NNN} + \left(\psi_{1s}\frac{\partial}{\partial N} - \psi_{1N}\frac{\partial}{\partial s}\right)\frac{\psi_{1N}}{r_0^j} = -r_0^j\, U_1(s,0)\, U_{1s}(s,0), \tag{4.9a}$$

$$\psi_1(s,0) = \psi_{1N}(s,0) = 0, \tag{4.9b}$$

$$\psi_{1N}(s,\infty) = r_0^j\, U_1(s,0), \tag{4.9c}$$

and for the temperature field

$$\sigma^{-1} t_{1NN} + \frac{1}{r_0^j}(\psi_{1s} t_{1N} - \psi_{1N} t_{1s}) = \tfrac{1}{2}m^2\left[\frac{1}{r_0^j}\left(\psi_{1N}\frac{\partial}{\partial s} - \psi_{1s}\frac{\partial}{\partial N}\right) - \frac{\partial^2}{\partial N^2}\right]\left(\frac{\psi_{1N}}{r_0^j}\right)^2, \tag{4.10a}$$

$$t_1(s,0) = T_w(s), \quad t_{1N}(s,0) = 0, \quad \text{or the like}, \tag{4.10b}$$

$$t_1(s,\infty) = H_1(0) - \tfrac{1}{2}m^2 U_1^2(s,0). \tag{4.10c}$$

The corresponding second-order problems are

$$\begin{aligned}\psi_{2NNN} &+ \left(\psi_{1s}\frac{\partial}{\partial N} - \psi_{1N}\frac{\partial}{\partial s}\right)\frac{\psi_{2N}}{r_0^j} + \left(\psi_{2s}\frac{\partial}{\partial N} - \psi_{2N}\frac{\partial}{\partial s}\right)\frac{\psi_{1N}}{r_0^j} \\ &= r_0^j\frac{\partial}{\partial s}\kappa\left[NU_1^2(s,0) + \int_N^\infty\left\{U_1^2(s,0) - \frac{\psi_{1N}^2(s,N)}{r_0^{2j}}\right\}dN\right] - \kappa\left(N\psi_{1NNN} + \psi_{1NN} + \frac{\psi_{1s}\psi_{1N}}{r_0^j}\right) \\ &\quad + j\left[\frac{\cos\theta}{r_0}\psi_{1NN} + \left(\psi_{1s}\frac{\partial}{\partial N} - \psi_{1N}\frac{\partial}{\partial s}\right)N\frac{\cos\theta}{r_0}\frac{\psi_{1N}}{r_0^j} - N\frac{\cos\theta}{r_0} r_0^j\, U_1(s,0)\, U_{1s}(s,0)\right] \\ &\quad - r_0^{2j} B_1'(0)\, V_2(s,0) - r_0^j\frac{d}{ds}[U_1(s,0)\, U_2(s,0)],\end{aligned} \tag{4.11a}$$

$$\psi_2(s,0) = \psi_{2N}(s,0) = 0, \tag{4.11b}$$

$$\psi_{2N}(s,N) \sim N r_0^j\left[\left(j\frac{\cos\theta}{r_0} - \kappa\right) U_1(s,0) + r_0^j B_1'(0)\right] + r_0^j\, U_2(s,0), \tag{4.11c}$$

and

$$\sigma^{-1}t_{2NN}+\frac{1}{r_0^j}(\psi_{1s}t_{2N}-\psi_{1N}t_{2s})$$
$$=\frac{1}{r_0^j}\left(\psi_{2N}\frac{\partial}{\partial s}-\psi_{2s}\frac{\partial}{\partial N}\right)\left[t_1+\tfrac{1}{2}m^2\left(\frac{\psi_{1N}}{r_0^j}\right)^2\right]$$
$$-\left(\kappa+j\frac{\cos\theta}{r_0}\right)\frac{\partial}{\partial N}N\frac{\partial}{\partial N}\left[\sigma^{-1}t_1+\tfrac{1}{2}m^2\left(\frac{\psi_{1N}}{r_0^j}\right)^2\right]+m^2\kappa\frac{\partial}{\partial N}\left(\frac{\psi_{1N}}{r_0^j}\right)^2$$
$$+m^2\left[\frac{\partial^2}{\partial N^2}+\frac{1}{r_0^j}\left(\psi_{1s}\frac{\partial}{\partial N}-\psi_{1N}\frac{\partial}{\partial s}\right)\right]\frac{\psi_{1N}}{r_0^{2j}}\left(jN\frac{\cos\theta}{r_0}\psi_{1N}-\psi_{2N}\right), \quad (4.12a)$$

$$t_2(s,0)=0,\quad t_{2N}(s,0)=0,\quad \text{or the like}, \quad (4.12b)$$

$$t_2(s,N)\sim NU_1(s,0)\,[H_1'(0)\,r_0^j-m^2B_1'(0)\,r_0^j+m^2\kappa U_1(s,0)]$$
$$+[H_1'(0)\,\Psi_2(s,0)-m^2U_1(s,0)\,U_2(s,0)]. \quad (4.12c)$$

The skin friction is given by

$$\tau=(\rho U_r^2)\,R^{-\frac{1}{2}}r_0^{-j}[\psi_{1NN}(s,0)+R^{-\frac{1}{2}}\psi_{2NN}(s,0)+\ldots], \quad (4.13)$$

and the heat transfer from the surface by

$$q=-(kT_r/L)\,R^{\frac{1}{2}}[t_{1N}(s,0)+R^{-\frac{1}{2}}t_{2N}(s,0)+\ldots]. \quad (4.14)$$

### 4.4. *Decomposition of second-order problem*

Because the second-order boundary-layer problem is linear we can, following the suggestion of Rott & Lenard (1959), subdivide it into a number of simpler problems, each of which has a clear physical interpretation. To this end, all non-homogeneous terms have been written on the right.

Consider the problem (4.11) for the velocity field. The non-homogeneous terms fall into two main categories. Those proportional to $\kappa$ and $j$ arise from curvature of the body surface. The remainder arise from the displacement effect of the first-order boundary layer upon the outer flow, which is reflected in second-order changes in the pressure and temperature at the outer edge of the boundary layer, given by (3.14).

It is sometimes convenient to subdivide further these two categories, although this secondary decomposition is somewhat arbitrary. Of the curvature terms, those in $\kappa$ are present in either plane or axisymmetric flow, and represent the effects of *longitudinal curvature*. Those in $j$ must be added in axisymmetric flow, and are therefore said to arise from *transverse curvature*. In the terms associated with displacement of the outer flow, $U_2(s,0)$ is the change induced in the speed at the outer edge of the boundary layer; we shall call its effect that of *displacement speed*. Terms proportional to $B_1'(0)$ represent, according to (3.9*b*), the direct effects of *external vorticity* in the basic oncoming stream. (Vorticity has also an indirect effect in that it influences every component of the solution, beginning with the basic inviscid flow. The linear effect isolated here simply shows the effect of external vorticity if the outer surface speed is held fixed.)

The second-order stream function can therefore be subdivided into four components by setting

$$\psi_2=\psi_2^{(l)}+j\psi_2^{(t)}+B_1'(0)\,\psi_2^{(v)}+\psi_2^{(d)} \quad (4.15a)$$

(with corresponding expressions for $u_2$, $v_2$ and $p_2$). Here the superscripts identify respectively the contributions of longitudinal curvature, transverse curvature, external vorticity, and displacement speed.

In the problem for the temperature field, the first term on the right of (4.12*a*) may be interpreted physically as the change in convected stagnation enthalpy that results from the modification of streamlines. Substituting (4.15*a*) suggests that $t_2$ may be subdivided just as $\psi_2$ was. However, a new effect arises from the terms proportional to $H_1'(0)$, which represent the effect of a *stagnation enthalpy gradient* in the oncoming stream. Thus the temperature is separated into five components by setting

$$t_2 = t_2^{(l)} + jt_2^{(t)} + B_1'(0)\,t_2^{(v)} + H_1'(0)\,t_2^{(H)} + t_2^{(d)}. \tag{4.15b}$$

(Components proportional to $B_2(0)$ and $H_2(0)$ would also appear if the oncoming stream depended on Reynolds number.) Substituting these decompositions into the second-order problem leads to an individual problem for each of the five effects enumerated.

The individual matching conditions can, like their first-order counterparts (4.2*c*), be imposed at $N = \infty$, provided that $N$ is first removed by differentiation if it appears explicitly. Thus in the problem for displacement speed, (4.11*c*) gives immediately

$$\psi_{2N}^{(d)}(s, \infty) = r_0^j\, U_2(s, 0), \tag{4.16}$$

whereas for external vorticity one obtains first

$$\psi_{2N}^{(v)}(s, N) \sim Nr_0^{2j} + o(1) \quad \text{as} \quad N \to \infty. \tag{4.17a}$$

Now the fact expressed by the second term, that there is no term of order unity, follows from the first term and the differential equation (because of the substitution therein of (4.5)). Therefore the condition may be simplified to

$$\psi_{2NN}^{(v)}(s, \infty) = r_0^{2j}. \tag{4.17b}$$

For an insulated wall and Prandtl number unity, the individual energy equations for the problems of transverse curvature, external vorticity and displacement speed have the integral

$$t_2 + m^2 u_1 u_2 = 0. \tag{4.18}$$

An alternative way of separating vorticity and displacement effects is of interest. The penultimate term in (4.6*b*) or (4.11*a*), which represents the pressure gradient arising from interaction of displacement and vorticity, may be regarded as a displacement rather than a vorticity effect. Adding it to the problem for displacement speed gives what we shall call the problem for the effect of *displacement pressure*. Then external vorticity retains only a kinematic role, through its appearance in the matching condition (4.7*c*) or (4.11*c*); we shall refer to this as the *kinematic effect of external vorticity*. The component matching conditions (4.16) and (4.17) remain unaltered.

Most previous investigators of external vorticity have adopted (at least tacitly) this alternative subdivision involving displacement pressure rather than displacement speed, so that they calculate only the kinematic effect of vorticity. Unfortunately, as will be discussed in detail in Part 2, those investigators who have also considered displacement effects have usually overlooked the additional effect of vorticity in inducing a pressure perturbation.

For example, Hayes (1956) has suggested that in the problem for external vorticity the matching condition for the stream function is to be found by eliminating $n$ between $\psi$ and $\psi_n$ in the basic inviscid flow. This leads, with (4.17$a$) assumed, to

$$\psi_{2N}^{(d)}(s,\infty) = B_1'(0)\, r_0^j\, \Psi_2(s,0)/U_1(s,0),$$

instead of (4.16) for the displacement effect. Comparison with (3.14$a$) shows that this condition corresponds to neglecting the second-order pressure perturbation $P_2(s, 0)$ at the edge of the boundary layer. Thus Hayes' condition is correct for the kinematic effect of vorticity, but incorrect for the remaining effect of displacement unless the induced pressure perturbation $P_2(s, 0)$ happens to vanish.

The effect of displacement speed is much harder to calculate than the others, because it alone requires the determination of the outer flow due to displacement. (The effects of external vorticity and stagnation enthalpy gradient involve $V_2$ and $\Psi_2$, but only at the surface, where they are given in terms of the classical boundary-layer solution by (3.27).) The displacement-speed effect is therefore global, whereas the others are local. Only through it does the elliptic nature of the Navier–Stokes equations reassert itself in the boundary layer, having been suppressed in Prandtl's equations, which are parabolic and therefore permit no upstream influence.

### 4.5. *Change of co-ordinates*

The outer solution is independent of the choice of co-ordinate system (so that the equations governing it can be written in vector form). On the other hand, the classical boundary-layer solution is known to depend upon the choice of co-ordinates, in a manner that has been studied by Kaplun (1954). Because it is often convenient to use boundary-layer co-ordinates other than the present $(s, n)$ system, we extend some of Kaplun's results to second order.

Consider a general co-ordinate system $(\xi, \eta)$ that need not be orthogonal, but for convenience is such that the body is described by a co-ordinate line

$$\eta = \eta_0 = \text{const.}$$

The transformation $\xi = \xi(s,n)$, $\eta = \eta(s,n)$ is assumed regular, so that near the body

$$\left.\begin{aligned} \xi &= \xi(s,0) + n\xi_n(s,0) + O(n^2), \\ \eta &= \eta_0 + n\eta_n(s,0) + \tfrac{1}{2}n^2\eta_{nn}(s,0) + O(n^3). \end{aligned}\right\} \qquad (4.19)$$

Introduce the magnified inner variable $H$ related to $\eta$ as $N$ is to $n$:

$$H = R^{\frac{1}{2}}(\eta - \eta_0). \qquad (4.20)$$

Corresponding to the inner expansion (3.18) in the $(s, n)$-co-ordinates, we seek an inner expansion in the new co-ordinates of the form

$$\left.\begin{aligned} u &\sim \tilde{u}_1(\xi, H) + R^{-\frac{1}{2}}\tilde{u}_2(\xi, H) + \ldots, \\ v &\sim R^{-\frac{1}{2}}\tilde{v}_1(\xi, H) + R^{-1}\tilde{v}_2(\xi, H) + \ldots, \end{aligned}\right\} \qquad (4.21)$$

and similarly for $p$, $T$, and $\psi$. Expressing $(\xi, H)$ here in terms of $(s, N)$ by means of (4.19) and (3.17), expanding in Taylor series for large $R$, and equating to (3.18) gives the desired result:

$$u_1(s,N) = \tilde{u}_1[\xi(s,0), N\eta_n(s,0)], \tag{4.22a}$$

$$u_2(s,N) = \tilde{u}_2[\ ] + N\xi_n(s,0)\,\tilde{u}_{1\xi}[\ ] + \tfrac{1}{2}N^2\eta_{nn}(s,0)\,\tilde{u}_{1H}[\ ], \tag{4.22b}$$

with completely analogous relations for $v_1$ and $v_2$, $t_1$ and $t_2$, and $\psi_1$ and $\psi_2$.

The first of these is Kaplun's correlation theorem, which relates the classical boundary-layer solutions in any two co-ordinate systems, and the second is its counterpart in second-order boundary-layer theory. The boundary-layer problems in the new co-ordinate system are found by substituting (4.22) into (4.8)–(4.12), and using the transformation (4.19) and its inverse.

Kaplun has shown how, from the flow due to displacement thickness, one can determine certain *optimal co-ordinates* in which the classical boundary-layer solution is valid also in the outer flow. This idea could be extended to the second approximation. Alternatively, a single uniformly-valid composite expansion can, if required, be formed from the inner and outer expansions (cf. Kaplun & Lagerstrom 1957).

### 4.6. *Behaviour far downstream*

The present analysis gives an asymptotic solution for large Reynolds number that is uniformly valid for finite distances from the nose of the body. However, it may become invalid far downstream in some cases. Thus Prandtl's boundary-layer approximation becomes invalid as $s \to \infty$ if the boundary-layer thickness $\delta(s)$ grows faster than either the longitudinal or transverse radius of curvature of the body; that is, if either $\delta(s)\,\kappa(s)$ or $\delta(s)/r_0(s)$ is unbounded. The first possibility does not ordinarily arise in problems of interest; for example, for a parabola or paraboloid in a uniform stream, $\delta\kappa$ vanishes like $s^{-1}$. However, the second possibility may be realized in axisymmetric flow, as for uniform flow past a body that grows more slowly than a paraboloid.

The remedy in the first approximation is to replace (4.1$a$) and (4.1$b$) by Millikan's (1932) boundary-layer equations

$$(ru_1)_s + (rv_1)_N = 0, \tag{4.23a}$$

$$u_1 u_{1s} + v_1 u_{1N} - u_{1NN} - (r_N/r)\,u_N = U_1(s,0)\,U_{1s}(s,0). \tag{4.23b}$$

These are valid far downstream because the radius $r(s,n) = r_0(s) + n\cos\theta(s)$ has not been approximated by its value $r_0$ at the surface. Thus the effect of transverse curvature, which can grow from second to first order, is included in the leading approximation. With this as a basis, one could revise the preceding analysis to construct either a uniformly-valid inner expansion, or a supplementary one for large $s$. The leading term has been studied in the case of a circular cylinder by Seban & Bond (1951), Glauert & Lighthill (1955), and Stewartson (1955, 1957).

The displacement effect is uniformly small if $\delta\kappa$ remains bounded, but external vorticity leads to non-uniformity downstream. The reason is that the vorticity in the boundary layer is attenuated downstream by diffusion, so that eventually it is not large compared with the external vorticity. Again the remedy would be to include the effect in the first approximation. This can in principle be accomplished by repeating the matching that led to (3.25) with the understanding that $N$ may be large. The result is tractable, however, only for the semi-infinite flat plate, which has been treated asymptotically by Ting (1960).

This work draws heavily on discussions and correspondence with many colleagues, including S. Goldstein, J.-P. Guiraud, W. D. Hayes, S. Kaplun, N. Kemp, P. A. Lagerstrom, K. Mangler, R. Mark, N. Rott and J. T. Stuart. The earlier stages of the work were carried out at the Lockheed Missiles and Space Company.

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